Symmetries in Physical Theories

Undergraduate Research Thesis

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Abstract

We review the application of Noether's theorem to global internal and external symmetries. Applying the gauge procedure of Yang-Mills, we exhibit the emergence of physical interactions as a consequence of gauge symmetries. This procedure is carried out for U(1), SU(2) and Poincaré transformations. We connect Poincaré gauge theory with the corresponding theory of gravity, Einstein-Cartan-Sciama-Kibble theory, and examine the physical consequences of a torsionful affine connection.

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1 Introduction

Symmetry is the foundation of modern physics. Using symmetries, one can simplify a problem or predict what structure the theory should have. More specifically, in high energy physics, one often starts with a specific Lagrangian that they wish to use to describe a certain experimental phenomenon. The choice of a Lagrangian determines the equations that govern the motion it seeks to describe. Given the infinite choices of Lagrangians, it is a daunting task to construct one without a set of rules to govern the form it can take.

The resolution to this difficulty lies in symmetries. Historically, using symmetry in physics is not a new idea. Emmy Noether (Noether, 1918) postulated that symmetries have physical consequences that one can observe. There exists certain symmetries we have found to be experimentally respected in most theories we construct. Examples include Lorentz invariance, translation invariance, and phase invariance. In Section 2, we explore the consequences of requiring theories to respect these symmetries globally. The next natural step is to require that these symmetries are respected locally and observe what kind of physical predictions this requirement makes. With a restriction to internal symmetries, this is the focus of Section 3. In Section 4, we explore a theory that respects Poincaré invariance locally. In general, we seek to illustrate the predictive power of requiring symmetric Lagrangians and observe that many fundamental theories can be interpreted as byproducts of these symmetries.

2 Global Symmetries

2.1 Noether's Theorem

At the heart of the connection between symmetries and the physical world is Emmy Noether's (first) theorem. It connects the conservation of measurable quantities with mathematical patterns of nature (Noether, 1918). Although we recount the classical version here, there is a quantum version of Noether's theorem that applies to correlation functions instead of observables (Takahashi, 1957).

Theorem (Noether)**.** *Suppose ϕ is a classical complex scalar field, and a Lagrangian density $\mathscr{L}$ is a function of ϕ and its related variables: $\mathscr{L}(\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*)$. Also suppose we have a smooth, continuous transformation*

$$\phi \to \phi' \tag{2.1}$$

$$x \to x' \tag{2.2}$$

that preserves the action up to a total derivative

$$S[\phi] \to S[\phi'] = S[\phi] + \int d^4x\, \partial_\mu K^\mu \tag{2.3}$$

Then the quantity

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^* \tag{2.4}$$

is conserved up to a boundary term K^μ, where

$$\delta\phi \equiv \phi'(x') - \phi(x)$$
$$\delta\phi^* \equiv \phi'^*(x') - \phi^*(x)$$

Proof. Since $S[\phi]$ is invariant under (2.1) and (2.2), we have

$$\begin{aligned}\delta S &= S[\phi'] - S[\phi] \\ &= \int d^4x \left(\frac{\partial \mathscr{L}}{\partial\phi}\delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right) + \int d^4x \left(\frac{\partial \mathscr{L}}{\partial\phi^*}\delta\phi^* + \frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi^*)}\delta(\partial_\mu\phi*)\right) \\ &= \int d^4x \left[\left(\frac{\partial \mathscr{L}}{\partial\phi} - \partial_\mu\frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \left(\frac{\partial \mathscr{L}}{\partial\phi^*} - \partial_\mu\frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi^*)}\right)\delta\phi^* + \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^*\right)\right]\end{aligned} \tag{2.5}$$

where I have Taylor-expanded S and kept only the linear terms. Since ϕ is on-shell, the first two terms of (2.5) vanish–these are the Euler-Lagrange equations for a complex scalar field. We are left with

$$\begin{aligned}\int d^4x\partial_\mu K^\mu &= \int d^4x\, \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^*)}\delta\phi^*\right) \\ \Rightarrow 0 &= \int d^4x\, \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^*)}\delta\phi^* - K^\mu\right)\end{aligned}$$

Since we may integrate over any region, we find

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^*)}\delta\phi^* - K^\mu\right) = 0$$

and thus

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^*)}\delta\phi^* - K^\mu$$

is conserved. Here, K^μ is the boundary term. □

Although a complex scalar field is specified in this proof, Noether's theorem applies to any action that has a symmetry as outlined above (Noether, 1918). Additionally, the symmetry need not be global for Noether's theorem to apply. However, as will be explored in §3, requiring invariance under a spacetime dependent symmetry requires a modification of the corresponding globally invariant theory. It is also crucial that the action only vary by a total derivative. If the action varies by something other than a total derivative, there is no implied conservation law (e.g. symmetry-breaking background fields).

2.2 Internal and External Spaces

For any field ψ, that maps elements of a vector space V to a vector space U,

$$\psi : V \to U \tag{2.6}$$

we say a symmetry is **internal** if the corresponding transformation acts on the internal space U. Similarly, a symmetry is **external** if the corresponding transformation acts on the external space V (usually in physics, V is spacetime or $\mathbb{R}^4$).

A global symmetry is a type of symmetry in which the variation is applied homogeneously to all values of x (and consequently, all values of the field). For a global internal, infinitesimal[1] variation we have

$$\phi(x) \to \phi'(x) = \phi(x) + i\alpha^a t^a \phi(x) \tag{2.7}$$

where the α^as are x independent, small parameters. We may also write this transformation non-infinitesimally as an exponential map:

$$\phi(x) \to \phi'(x) = e^{i\alpha^a t^a}\phi(x) \tag{2.8}$$

The t^a operators are the generators of the corresponding Lie group of the symmetry. These operators form the Lie algebra of the group,

$$\left[t^a, t^b\right] = if^{abc}t^c \tag{2.9}$$

where the completely antisymmetric f^{abc}s are the structure constants of the group. Global symmetry groups can be either Abelian or non-Abelian. These groups are quite general and can be used to explain theories built upon internal or external symmetries. Excluding the boundary term, we write the conserved current of a global internal symmetry as

$$J^a_\mu = \frac{\partial \mathscr{L}}{\partial(\partial^\mu \phi_i)} t^a_{ij}\phi_j \tag{2.10}$$

[1] Although discrete transformations are important in particle physics, we deal only in smooth, continuous transformations in the present work.

The conserved *Noether charges* are therefore

$$Q^a = \int d^3x J_0^a(x) \tag{2.11}$$

These charges also form a representation of the symmetry group (see Bañados and Reyes, 2016),

$$\left[Q^a, Q^b\right] = i f^{abc} Q^c \tag{2.12}$$

2.3 U(1)

The first symmetry we require theories to have is an invariance under a U(1) transformation. The fundamental representation of U(1) is the group of complex numbers with modulus 1. Complex numbers satisfying $|z| = 1$ are unitary as

$$\forall z \in U(1) : z^\dagger z = z^* z = |z|^2 = 1 \tag{2.13}$$

Suppose we have a complex scalar field ϕ. We can then write our prototype U(1)-invariant Lagrangian as

$$\mathscr{L} = \partial_\mu \phi^* \partial^\mu \phi \tag{2.14}$$

We can represent group elements by the transformation

$$\phi \to \phi' = e^{i\alpha}\phi \tag{2.15}$$

or infinitesimally,

$$\phi \to \phi + i\alpha\phi \tag{2.16}$$

Where $\alpha \in \mathbb{R}$ is a free parameter. Therefore, the generator of the U(1) symmetry is[2] 1. A U(1) symmetry is physically motivated by the conservation of probability. For example, in quantum mechanics, the probability of a interaction is related to the modulus squared of the inner product of two states. Observables (e.g. cross section σ and decay rate $\mathit{\Gamma}$) are then calculated from this inner product

$$\sigma, \mathit{\Gamma} \propto |\langle f|i\rangle|^2 \tag{2.17}$$

where $|i\rangle, |f\rangle$ are the initial and final states of some interaction process. These observables are invariant under a U(1) transformation sending $|i\rangle \to e^{i\alpha}|i\rangle$. Therefore it is natural for our Lagrangian to be invariant under U(1) phase transformations.

Via (2.4), we can observe the conserved current to be

$$J_\mu = i\left[(\partial_\mu \phi^*)\phi - \phi^*(\partial_\mu \phi)\right] \tag{2.18}$$

The generator of the U(1) symmetry group is then

$$Q = i \int d^3x \left(\dot{\phi}^* \phi - \phi^* \dot{\phi}\right) \tag{2.19}$$

If we quantize the theory by writing ϕ and ϕ^* as operators in terms of creation $(a^\dagger, b^\dagger)$ and annihilation (a, b) operators,

$$\widehat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \left(\widehat{a}_{\vec{k}} e^{-ik\cdot x} + \widehat{b}_{\vec{k}}^\dagger e^{ik\cdot x}\right) \tag{2.20}$$

$$\widehat{\phi}^*(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \left(\widehat{a}_{\vec{k}}^\dagger e^{ik\cdot x} + \widehat{b}_{\vec{k}} e^{-ik\cdot x}\right) \tag{2.21}$$

[2] Here we choose the normalization $\mathrm{tr}(\mathrm{t^a t^b}) = 1$ since there is only 1 generator.

then (2.19) becomes[3]

$$Q = \int \frac{d^3k}{(2\pi)^3 2E_k} \left(\widehat{a}^\dagger_{\vec{k}} \widehat{a}_{\vec{k}} - \widehat{b}^\dagger_{\vec{k}} \widehat{b}_{\vec{k}} \right) \tag{2.22}$$

$$= \int \frac{d^3k}{(2\pi)^3 2E_k} \left(\widehat{N}_p - \widehat{N}_{\bar{p}} \right) \tag{2.23}$$

where $\widehat{N}_p$ and $\widehat{N}_{\bar{p}}$ represent the particle and antiparticle number operators respectively. This gives the nice interpretation that U(1) invariance implies total particle number conservation[4].

2.4 SU(2) Isospin

The next natural mathematical extension from a U(1) symmetry is a general SU(N) symmetry. Here we will consider one of the first extensions of U(1) theory-isospin. Isospin is a type of flavor symmetry in which multiplets represent particle species that transform into each other under the action of SU(2). The corresponding Lie group to isospin is SU(2). Historically, Murray Gell-Mann developed flavor symmetry as a way of organizing the particle zoo in the 1960s (Gell-Mann, 1961). The full flavor symmetry he considered was SU(3) where the the up, down, and strange quarks form an SU(3) triplet (Gell-Mann, 1961). Although the more exact symmetry is SU(3) color in the Standard Model, SU(3) flavor symmetry is useful in classifying hadronic states and calculating scattering amplitudes.

Analogous to angular momentum in quantum mechanics, we can represent isospin states with an isospin quantum number and its projection. We have the following isospin operators and their eigenvalues:

$$\begin{aligned} &I^2 : I(I+1),\ I = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \\ &I_3 : -I, -I+1, \dots, I \end{aligned}$$

For example we can represent the proton and neutron as an isospin doublet

$$\begin{pmatrix} p = |\frac{1}{2}, \frac{1}{2}\rangle \\ n = |\frac{1}{2}, -\frac{1}{2}\rangle \end{pmatrix}$$

or pions as an isospin triplet

$$\begin{pmatrix} \pi^+ = |1, 1\rangle \\ \pi^0 = |1, 0\rangle \\ \pi^- = |1, -1\rangle \end{pmatrix}$$

etc. Each of these multiplets form a representation of the isospin group. We will take the up and down quarks to be an isospin doublet,

$$\psi(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}$$

where $u(x), d(x)$ are spinors and $\psi(x)$ is a Dirac spinor. Since quarks are fermions, we take our Lagrangian to be

$$\mathscr{L}_{\text{Isospin}} = \overline{\psi}_i [i\partial\!\!\!/ - m] \psi_i \tag{2.24}$$

where $i = 1, 2$. For a global SU(2) transformation

$$\psi \to \psi' = \exp\left(-i\frac{\vec{\sigma}}{2} \cdot \vec{\theta}\right)\psi \equiv T\psi \tag{2.25}$$

[3]There is also an infinite energy density term related to renormalization but we ignore this term for our purposes (see Casimir and Polder, 1948).

[4]This interpretation becomes tricky for field theories with nonzero interacting Lagrangians. Interacting theories are usually quantized via path integrals and thus do not have concrete expressions for ψ in terms of creation and annihilation operators.

where $\vec{\theta}$ represents the 3 free parameters of SU(2). Clearly (2.24) is invariant under (2.25) since $\partial\!\!\!/ T = 0$. We can therefore write down the conserved isospin current

$$j^i_\mu = \overline{\psi}\gamma_\mu \frac{\sigma^i}{2}\psi$$

After quantizing the SU(2) theory, the interpretation of this current is similar to the U(1) case. When considering the full theory of the electroweak interactions this current becomes important when discussing chiral symmetry breaking, which is still not fully understood perturbatively [5] (Cheng and Li, 2000).

2.5 Translational Symmetry

Although internal symmetries are interesting, external symmetries are much easier to see in nature. For example, shifting an experiment by a small amount and repeating it to find unchanged results is a way to exhibit translational invariance. Suppose we have some Lagrangian $\mathscr{L}$ that is invariant under spacetime translations,

$$x^\mu \to x^{\mu\prime} = x^\mu + \epsilon^\mu. \tag{2.26}$$

Then we may write the variation of the Lagrangian as

$$\begin{aligned} \delta\mathscr{L} &= \mathscr{L}' - \mathscr{L} \\ &= -\epsilon^\mu(\partial_\mu \mathscr{L}) \\ &= -\epsilon^\nu \partial_\mu(\delta^\mu{}_\nu \mathscr{L}) \end{aligned} \tag{2.27}$$

On the other hand, we may write

$$\begin{aligned} \delta\mathscr{L} &= \frac{\partial\mathscr{L}}{\partial\phi}\delta\phi + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) \\ &= \left[\frac{\partial\mathscr{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right)\right]\delta\phi + \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) \\ &= \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}(-\epsilon^\nu\partial_\nu\phi)\right) \\ &= -\epsilon^\nu\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi\right) \end{aligned} \tag{2.28}$$

Equating (2.27) and (2.28) reveals

$$T^{\mu\nu} = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathscr{L}, \tag{2.29}$$

The energy-momentum (stress energy) tensor, is conserved. Note that the energy-momentum tensor is a rank-2 tensor because there is a conserved quantity for every direction in spacetime according to the variation ϵ^μ. Therefore, there are also 4 generators of the translation symmetry given by

$$Q^\mu = \int d^3x\, T^{0\mu}.$$

From the definition of the energy-momentum tensor we readily identify Q^μ as the 4-momentum of the particle,

$$\begin{aligned} P^\mu &= \int d^3x\, T^{0\mu} \\ &= \int d^3x \left(\frac{\partial\mathscr{L}}{\partial(\partial_0\phi)}\partial^\mu\phi - \eta^{0\mu}\mathscr{L}\right) \end{aligned} \tag{2.30}$$

[5] There are non-perturbative approaches to chiral symmetry breaking that have shown chiral symmetry breaking, such as lattice Quantum Chromodynamics.

We may also note that T^{00} is the Legendre transform of the Lagrangian, which is, by definition, the Hamiltonian density, $\mathscr{H}$:

$$
\begin{aligned}
T^{00} &= \frac{\partial \mathscr{L}}{\partial(\partial_0\phi)}\partial^0\phi - \mathscr{L} \\
&= \frac{\partial \mathscr{L}}{\partial\dot{\phi}}\dot{\phi} - \mathscr{L} \equiv \mathscr{H}
\end{aligned} \tag{2.31}
$$

In this way, a global translational symmetry implies the conservation of energy and momentum. Note that the form of $\mathscr{L}$ does not matter for energy and momentum to be conserved if translational invariance is observed. This is ultimately because external symmetries do not depend on the particle content of the theory.

2.6 Lorentz Symmetry

Another external symmetry is invariance under Lorentz transformations. Lorentz invariance is an important physical consequence of Special Relativity (see Misner et al., 1973). It states there is no preferred frame in which to conduct experiments. Before outlining the consequences of Lorentz invariance, we first define an infinitesimal Lorentz transformation. Consider a one-parameter family of Lorentz transformations (see Misner et al., 1973)

$$
\begin{aligned}
&\{\Lambda : \mathbb{R} \to \mathrm{SO}(1,3)\} \\
&\tau \to \Lambda(\tau)
\end{aligned} \tag{2.32}
$$

where we define $\Lambda(0) \equiv \mathbb{I}_{4\text{x}4}$. Since $\{\Lambda(\tau)\}$ consists of Lorentz transformation, we require

$$
\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^{\alpha}{}_{\mu}(\tau)\Lambda^{\beta}{}_{\nu}(\tau) \tag{2.33}
$$

Now expand (2.33) around the origin,

$$
\begin{aligned}
\frac{d}{d\tau}\left(\eta_{\alpha\beta}\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\right)|_{\tau=0} &= 2\eta_{\alpha\beta}\Lambda'^{(\alpha}{}_{\mu}\Lambda^{\beta)}{}_{\nu}|_{\tau=0} \\
&= 2\eta_{\alpha(\nu}\Lambda'^{\alpha}{}_{\mu)} \\
&= 0
\end{aligned} \tag{2.34}
$$

We define a matrix ω with components $\omega^{\alpha}{}_{\mu} = \Lambda'^{\alpha}{}_{\mu}(0) = \omega^{[\alpha}{}_{\mu]}$ and expand $\Lambda(\tau)$ near the origin,

$$
\Lambda^{\mu}{}_{\nu}(\tau) = \delta^{\mu}_{\nu} + \tau\omega^{\mu}{}_{\nu} \tag{2.35}
$$

where τ is now infinitesimally small. To first order, we then define a Lorentz transformation infinitesimally by

$$
x^{\mu} \to \Lambda^{\nu}{}_{\mu}x^{\mu} = x^{\nu} + \tau\omega^{\nu}{}_{\mu}x^{\mu}. \tag{2.36}
$$

Following the same reasoning in §2.5, we see the variation in the action is

$$
\delta S = -\int d^4x\, \partial_\mu \left(x^\alpha \left[\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu\phi\right)}\partial^\nu\phi - \eta^{\mu\nu}\mathscr{L} \right] \right) \tau\omega_{\nu\alpha} \tag{2.37}
$$

$$
= -\int d^4x\, \partial_\mu \left(x^\alpha T^{\mu\nu} \right) \tau\omega_{\nu\alpha} \tag{2.38}
$$

$$
= -\int d^4x\, \partial_\mu \left(x^{[\alpha} T^{\mu|\nu]} \right) \tau\omega_{\nu\alpha} \tag{2.39}
$$

where we use the fact that $\omega_{\alpha\nu}$ is antisymmetric. The conserved charges corresponding to the Lorentz symmetry are then

$$
M^{\mu\nu} = \int d^3x\, x^{[\mu}T^{0|\nu]} \tag{2.40}
$$

The Lie algebra of these generators is

$$[M_{\mu\nu}, M_{\rho\sigma}] = 4i\eta_{[\rho[\nu}M_{\mu]\sigma]} \tag{2.41}$$

We recognize the angular momentum operator by constructing

$$L_i = \frac{1}{2}\epsilon_{ijk}M^{jk} \tag{2.42}$$

$$= -i\left(\vec{x} \times \vec{\nabla}\right) \tag{2.43}$$

$M^{\mu\nu}$ can be thought of as a relativistic angular momentum, with the other generators of the Lorentz group, M_{0i}, being associated with boosts. Via (2.8), we can use the generators to write a Lorentz transformation as

$$\Lambda : x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu = e^{\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}}x^\nu \tag{2.44}$$

where $\omega_{\mu\nu}$ has 6 independent parameters corresponding to 3 boosts and 3 rotations in Minkowski space. According to Special Relativity, one must always have global Lorentz invariance. Therefore relativistic angular momentum must always be conserved. As with translational symmetries, the form that $\mathscr{L}$ takes here is irrelevant and therefore Lorentz invariance always implies conservation of relativistic angular momentum.

2.7 Poincaré Symmetry

Physically, Poincaré symmetry represents the freedom to choose one's frame and origin of an experiment. Mathematically, the Poincaré group is the semi-direct product of the group of translations and the full Lorentz group[6]:

$$\mathcal{P} = \mathbb{R}^{1,3} \ltimes O(1,3) \tag{2.45}$$

The generators of the Poincaré symmetry are P^μ and $M^{\mu\nu}$. The Lie algebra of the Poincaré group is given by (2.41) and

$$[P^\mu, P^\nu] = 0$$
$$[P^\mu, M^{\rho\sigma}] = ig^{\mu[\rho}P^{\sigma]}$$

Poincaré invariance corresponds to the conservation of 4-momentum and relativistic angular momentum. Poincaré transformations can be thought of as Lorentz transformations that do not preserve the origin (i.e. the inhomogenous Lorentz group). Correspondingly, all particle theories respect Poincaré invariance as far as we know (Chkareuli, 2017).

3 Internal Gauge Symmetries

In the previous section, all the symmetries were global symmetries–independent of spacetime. In this section, we consider the localization of those global symmetries – now denoted gauge symmetries. Gauge symmetries represent a redundancy of our physical system and do not correspond to a conservation law as in the global symmetry case. However, note that by taking a homogeneous transformation function, $\alpha(x) \equiv \alpha$, we recover the corresponding global symmetry and therefore a gauge symmetry automatically implies a globally conserved current. Instead we require Lagrangians to be gauge invariant so as to have a local description of our particle fields. There are four parts to gauging a global symmetry:

- Allow the parameters of the global symmetry to vary in spacetime.
- To compensate for this variation, introduce a covariant derivative along with a gauge connection ("compensating field").
- Determine the transformations of the gauge connection and field strength tensor, and construct the gauge invariant Lagrangian.
- Determine the field strength tensor via the commutator of the covariant derivative.

We first take a look at the gauge theory of electromagnetism.

[6]In the preceding section, I considered only proper Lorentz transformations, which belong to the subgroup SO(1,3) of O(1,3).

3.1 Spinor Electrodynamics

Consider the local U(1) transformation,

$$\psi \to e^{i\alpha(x)}\psi \tag{3.1}$$

where now we consider a four-component Dirac spinor ψ. A problem arises when we seek to compare our spinor field ψ at two different points x^μ and y^μ. Namely, we now have the freedom to choose a separate phase at each of these points. Therefore the discrepancy in the field values at these points,

$$|\psi(x) - \psi(y)| \to |e^{i\alpha(x)}\psi(x) - e^{i\alpha(y)}\psi(y)| \tag{3.2}$$

is now phase dependent; this is known as gauge freedom (Balachandran, 1994). Gauge freedom allows us to redefine our fields using a different coordinate system. Analogous to General Relativity, we want our description of the internal space to be independent of our specific phase choice at each point (Misner et al., 1973). In other words, we would like for theories to be gauge invariant.

Start with the free spinor field theory with the Dirac Lagrangian,

$$\mathscr{L}_{\text{Dirac}} = \overline{\psi}\left[i\partial\!\!\!/ - m\right]\psi \tag{3.3}$$

Under the action of (3.1), (3.3) becomes

$$\begin{aligned}\mathscr{L}_{\text{Dirac}} &\to e^{-i\alpha}\overline{\psi}[i\partial\!\!\!/ - m]\psi \\ &= \mathscr{L}_{\text{Dirac}} - \overline{\psi}\psi\partial\!\!\!/\alpha \end{aligned}\tag{3.4}$$

It is the spacetime dependence that spoils the U(1) invariance of (3.3). To restore U(1) invariance, we introduce a connection A_μ to allow comparison of field values in the internal space (without the need to choose a gauge). The gauge potential in spinor electrodynamics transforms like

$$A_\mu \to A_\mu - \partial_\mu\alpha(x) \tag{3.5}$$

The corresponding curvature to this U(1) connection is the familiar field strength tensor $F_{\mu\nu}$ where[7]

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$$

which can be seen from the commutator of (3.6). Since we are considering a U(1) symmetry, we also have $[A_\mu, A_\nu] = 0$. We can modify the Dirac Lagrangian to be gauge invariant by promoting our normal derivative to a covariant derivative, $\partial\!\!\!/ \to D\!\!\!\!/$, where

$$D_\mu = \partial_\mu + ieA_\mu = \partial_\mu - i(-e)A_\mu \tag{3.6}$$

and, under a (finite) gauge transformation

$$D_\mu \to e^{-i\alpha(x)}D_\mu e^{i\alpha(x)}$$

(3.3) then becomes the Lagrangian for Spinor Electrodynamics,

$$\mathscr{L} = \overline{\psi}[iD\!\!\!\!/ - m]\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{3.7}$$

The last term in (3.7) is to allow our vector potential to propagate. We can interpret this potential as a spin-1 massless gauge boson (Balachandran, 1994). This boson is known as the photon. Without the last term in (3.7), we would have no free photons (which would be in disagreement with our experience). We see then the first consequence of gauging a global symmetry is the necessity of introducing force mediators. In the case of a local U(1) symmetry, that force is electromagnetism. From (3.7), we see that there are physical consequences from adding the gauge potential. One of the more bizarre consequences of this interaction is vacuum polarization (Girotti et al., 1992). This phenomenon, obtained via renormalization and depicted in Figure 1, states the higher the energy of the interaction, the larger the charge of the electron. The interpretation is that the vacuum itself shields electric charge, which is unimaginable without the QED vertex.

Therefore, although gauge invariance does not correspond to a local conservation law, it does introduce interactions that describe physical processes; it is often useful to consider symmetries of fundamental importance to physical theories as a starting point for constructing their quantum counterparts.

[7] We assume a coordinate basis so that $[\partial_\mu, \partial_\nu] = 0$.

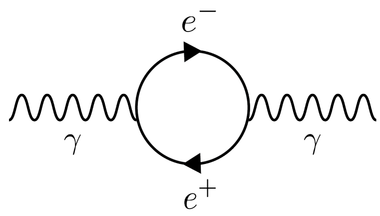


Figure 1: Vacuum polarization depicted heuristically as a correction to the U(1) gauge field propagator.

3.2 Yang-Mills Theory

Based on the success of gauging a U(1) symmetry, we now turn to gauging the theory of isospin. The procedure for a gauging a U(1) symmetry in the preceding section worked because U(1) is Abelian. A manifestation of the Abelian structure is the absence of a photon self-coupling. In a non-Abelian gauge theory, this is no longer true, as Yang and Mills showed in 1954 and as we will now see for the case of SU(2) isospin (Cheng and Li, 2000). Consider the isospin doublet

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix} \tag{3.8}$$

Under an SU(2) transformation, ψ transforms as

$$\psi \to \psi' = \exp\{i\frac{\vec{\sigma}}{2} \cdot \vec{\theta}\}\psi \equiv S(\theta)\psi \tag{3.9}$$

$$\overline{\psi} \to \overline{\psi}' = \overline{\psi}S^{-1}(\theta) \tag{3.10}$$

where $\vec{\theta} \in \mathbb{R}^3$ represent rotation parameters. Take $\vec{\theta} = \vec{\theta}(x)$ to be locally varying. If we transform the isospin Lagrangian (2.24) according to SU(2), we find

$$\begin{aligned}\mathscr{L}_{\text{Isospin}} &\to \overline{\psi}'[\partial\!\!\!/ - m]\psi' \\ &= \overline{\psi}[\partial\!\!\!/ - m]\psi + \overline{\psi}S^{-1}(\theta)\partial\!\!\!/ S(\theta)\psi \\ &= \mathscr{L}_{\text{Isospin}} + \overline{\psi}S^{-1}(\theta)\partial\!\!\!/ S(\theta)\psi\end{aligned} \tag{3.11}$$

Just as with QED, we face the problem of the last term in our transformed Lagrangian forcing us to make a choice of $\vec{\theta}$. To have a local description of the SU(2) ψ doublet, we must introduce a gauge connection to obtain a method of comparing our field at different points in the SU(2) space. However, unlike in the U(1) case, we must now introduce a nontrivial Lie-algebra valued connection. We can form the minimally-coupled covariant derivative

$$\mathbf{D}_\mu = \partial_\mu - ig\frac{\vec{\sigma}}{2} \cdot \vec{A}_\mu \equiv \partial_\mu - ig\mathbf{A}_\mu \tag{3.12}$$

where now g is an arbitrary coupling constant[8]. We would like the covariant derivative term to transform covariantly to ensure the isospin Lagrangian is invariant under (3.9). So we may write

$$\mathbf{D}'_\mu\psi' = S(\theta)\mathbf{D}_\mu\psi \tag{3.13}$$

From (3.13) we can derive a transformation law for $\mathbf{A}_\mu$. From the LHS of (3.13) we have

$$\begin{aligned}\mathbf{D}'_\mu\psi' &= (\partial_\mu - ig\mathbf{A}'_\mu)\psi' \\ &= (\partial_\mu - ig\mathbf{A}'_\mu)S(\theta)\psi \\ &= \partial_\mu S(\theta)\psi + S(\theta)\partial_\mu\psi - ig\mathbf{A}'_\mu S(\theta)\psi\end{aligned} \tag{3.14}$$

[8] g is arbitrary in the sense that it is to be determined from experiment.

and from the RHS we have

$$\begin{aligned}\mathbf{D}'_\mu\psi' &= S(\theta)\mathbf{D}_\mu\psi \\ &= S(\theta)(\partial_\mu - ig\mathbf{A}_\mu)\psi \\ &= S(\theta)\partial_\mu\psi - S(\theta)ig\mathbf{A}_\mu\psi \end{aligned} \tag{3.15}$$

so that

$$\begin{aligned} -ig\mathbf{A}'_\mu S(\theta) &= -S(\theta)ig\mathbf{A}_\mu - \partial_\mu S(\theta) \\ \Rightarrow \quad \mathbf{A}'_\mu &= S(\theta)\mathbf{A}_\mu S^\dagger(\theta) - \frac{i}{g}\partial_\mu S(\theta)S^\dagger(\theta) \end{aligned} \tag{3.16}$$

where we have used the fact that for an element $G \in$ SU(N), we have $G^{-1} = G^\dagger$. (3.16) defines the general transformation law for the gauge potential $\mathbf{A}_\mu$. By using the infinitesimal version of an SU(2) transformation[9],

$$\begin{aligned} S(\theta) &= 1 + i\frac{\vec{\sigma}}{2}\cdot\vec{\theta} \\ S^\dagger(\theta) &= 1 - i\frac{\vec{\sigma}}{2}\cdot\vec{\theta} \end{aligned}$$

we can define an infinitesimal version of (3.16):

$$\begin{aligned} \mathbf{A}'_\mu &= (1 + i\frac{\vec{\sigma}}{2}\cdot\vec{\theta})\mathbf{A}_\mu(1 - i\frac{\vec{\sigma}}{2}\cdot\vec{\theta}) - \frac{i}{g}\partial_\mu(i\frac{\vec{\sigma}}{2}\cdot\vec{\theta})(1 - i\frac{\vec{\sigma}}{2}\cdot\vec{\theta}) \\ &= \mathbf{A}_\mu - i[\mathbf{A}_\mu, \frac{\vec{\sigma}}{2}\cdot\vec{\theta}] + \frac{1}{g}\partial_\mu(\frac{\vec{\sigma}}{2}\cdot\vec{\theta}) \end{aligned} \tag{3.17}$$

where we have used the SU(2) algebra in the second line and dropped $\mathcal{O}(\theta^2)$ terms. Rewriting (3.17), we find the transformation law of the components of $\mathbf{A}_\mu$ to be

$$A'^i_\mu = A^i_\mu + \frac{1}{g}\partial_\mu\theta^i - \epsilon^{ijk}\theta^j A^k_\mu \tag{3.18}$$

Now that we know how our gauge fields $\mathbf{A}_\mu$ transform, we must construct the corresponding curvature tensor to allow these fields to propagate. From differential geometry (see Misner et al., 1973), we know

$$\begin{aligned} -ig\mathbf{F}_{\mu\nu} &= [\mathbf{D}_\mu, \mathbf{D}_\nu] \\ &= (\partial_\mu - ig\mathbf{A}_\mu)(\partial_\nu - ig\mathbf{A}_\nu) - (\partial_\nu - ig\mathbf{A}_\nu)(\partial_\mu - ig\mathbf{A}_\mu) \qquad (3.19) \\ &= -g^2\mathbf{A}_\mu\mathbf{A}_\nu - ig\mathbf{A}_\mu\partial_\nu - ig\partial_\mu\mathbf{A}_\nu + g^2\mathbf{A}_\nu\mathbf{A}_\mu + ig\mathbf{A}_\nu\partial_\mu + ig\partial_\nu\mathbf{A}_\mu - ig\mathbf{A}_\nu\partial_\mu + ig\mathbf{A}_\mu\partial_\nu \\ &= g^2[\mathbf{A}_\nu, \mathbf{A}_\mu] + ig\partial_\nu\mathbf{A}_\mu - ig\partial_\mu\mathbf{A}_\nu \qquad (3.20) \end{aligned}$$

where the last two terms in (3.19) arise from the fact that covariant derivatives act on fields not explicitly shown here and so produce Leibniz terms. Therefore

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu - ig[\mathbf{A}_\mu, \mathbf{A}_\nu] \tag{3.21}$$

or in terms of the field strength tensor components:

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - ig\epsilon^{ijk}A^j_\mu A^k_\nu \tag{3.22}$$

[9]Since we are working on a manifold M, our tensor fields act on smooth functions and thus absolve the need for explicit finite transformations. Instead we argue that these infinitesimal transformations may be combined to form finite transformations and we say our theory is general description of any transformation in our manifold M. This caveat applies to all theories in the present work.

which is clearly different from the U(1) case. This extra term can be attributed to the non-commutativity of the SU(2) generators–had they commuted, the last term in (3.21) would vanish. The last piece of information we need to complete the construction of a local SU(2) gauge theory is the transformation law for $\mathbf{F}_{\mu\nu}$ (or equivalently $F^i_{\mu\nu}$). Transforming (3.22), we find

$$\begin{aligned}F^i_{\mu\nu} &\to \partial_{[\mu}\left(A^i_{\nu]} + \frac{1}{g}\partial_{\nu]}\theta^i - \epsilon^{ijk}\theta^j A^k_{\nu]}\right) - ig\epsilon^{ijk}\left(A^j_\mu + \frac{1}{g}\partial_\mu\theta^j - \epsilon^{jlm}\theta^l A^m_\mu\right)\left(A^k_\nu + \frac{1}{g}\partial_\nu\theta^k - \epsilon^{kab}\theta^a A^b_\nu\right)\\ &= \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - \epsilon^{ijk}\theta^j\left(\partial_\mu A^k_\nu - \partial_\nu A^k_\mu - ig\epsilon^{klm}A^l_\mu A^m_\nu\right)\\ &= F^i_{\mu\nu} - \epsilon^{ijk}\theta^j F^k_{\mu\nu}\end{aligned} \tag{3.23}$$

From (3.23) we see $F^i_{\mu\nu}$ transforms in the adjoint representation of SU(2). We may write (3.23) non-infinitesimally as

$$\mathbf{F}_{\mu\nu} \to S(\theta)\mathbf{F}_{\mu\nu}S^\dagger(\theta) \tag{3.24}$$

In the Abelian case, $F_{\mu\nu}$ was already gauge invariant. In the non-Abelian case, this is no longer true. So now we must show our gauge field propagation term remains gauge invariant[10]:

$$\begin{aligned}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} = \mathrm{Tr}\left[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right] &\to \mathrm{Tr}\left[S\mathbf{F}_{\mu\nu}S^\dagger S\mathbf{F}^{\mu\nu}S^\dagger\right]\\ &= \mathrm{Tr}\left[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right]\end{aligned} \tag{3.25}$$

Now that we have all the ingredients for a gauge invariant isospin theory, we may write down the gauge invariant Lagrangian:

$$\mathscr{L}_{\text{Isospin}} = -\frac{1}{4}\mathrm{Tr}\left[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right] + \overline{\psi}\left[i\not{\mathbf{D}} - m\right]\psi \tag{3.26}$$

Via experiment (see Zyla, 2020 and de Divitiis et al., 2012), we know $m_u \neq m_d$ so m in 2-flavor space is

$$\begin{pmatrix} m_u & 0\\ 0 & m_d\end{pmatrix} \tag{3.27}$$

and so isospin symmetry is explicitly broken. However, when considering larger ensembles (i.e. baryons, nuclei), isospin is a useful effective field theory as only a small portion of baryonic mass comes from the mass of the up and down quarks (although this approximation is more badly broken when one includes additional quark doublets) (Patel, 1992). The Standard Model is the most famous example of a Yang-Mills theory. Its symmetry group is SU(3)× SU(2) × U(1). The more exact isospin theory is the Glashow-Weinberg-Salam (GSW) model of the electroweak interaction. In this more complicated theory, there is a still an underlying symmetry, albeit with a symmetry group of SU(2)⊗U(1) (Cheng and Li, 2000). Therefore the symmetry under SU(N) is the basis upon which most of modern particle physics is built. However, as the Standard Model is based on internal symmetries, one could argue it is incomplete based on the exclusion of external symmetries. We now turn to external gauge theories to understand why they are fundamentally different from theories akin to the Standard Model.

4 External Gauge Symmetries[11]

Just as we can gauge internal symmetries, perhaps more physically relevant are external gauge theories. At some level, internal gauge theories require intuition about the underlying symmetry, and cannot always be easily verified. For example, isospin was born out of convenience and not necessity and, despite the success of the electroweak theory, it followed in the wake of many unsuccessful variants (Weinberg, 1995). When considering external symmetries, it is much easier to verify that the symmetry holds and thus, there is more

[10]Note there is nothing significant about the placement (upper v. lower) of a gauge index as Lie group generators do not transform under Lorentz transformations.

[11]In this Chapter, G is explicitly included to exhibit dimensionful quantities.

motivation to consider external gauge theories as a starting point for a quantum field theory. Unfortunately, external gauge theories are often not renormalizable (Shomer, 2007) and therefore additional structure is theoretically needed to promote them to quantum field theories. However, we concern ourselves only with the classical gauge theories, which we will explore now.

4.1 Global and Local Frames

As mentioned in §2.2, our external space is usually spacetime. If we now consider a Lorentz symmetry, it becomes important to describe a particle locally in flat Minkowski space (with a non-holonomic or orthogonal basis) as well as in global curved spacetime (with a holonomic or coordinate basis). General coordinate transformations, $GL(2,\mathbb{R})$, do not admit a spinor representation (see Gu, 2006 and Datta, 1971). Therefore, to include fermions in our theory, we need a tetrad (or *vierbein*) field. To distinguish the local components from the global, we use the standard convention of Latin letters for local components $(a, b, \ldots)$ and Greek letters for global components $(\alpha, \beta, \ldots)$.

The tetrad serves as a spacetime-dependant map between the local and global coordinate descriptions of our particle via the coefficients of the non-holonomic dual basis[12]

$$\vartheta^a = e^a_\mu(x)\, \omega^\mu \tag{4.1}$$

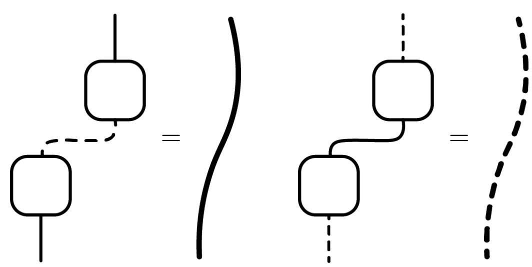
where the tetrad is denoted by $e^a_\mu(x)$. We can also think of the inverse tetrad as a map between the global and local bases (Gu, 2006)

$$e_a = e^\mu_a(x)\, e_\mu \tag{4.2}$$

We have the following identities

$$e^i_\mu e^\nu_i = \delta^\nu_\mu \qquad e^i_\mu e^\mu_j = \delta^i_j \tag{4.3}$$

or diagrammatically



For a general vector v^α, we have

$$v^\alpha = e^\alpha_i v^i \tag{4.4}$$

$$v^i = v^\alpha e^i_\alpha \tag{4.5}$$

Furthermore, we can relate the global and local metrics via the tetrad:

$$g_{\mu\nu}(x) = e_\mu \cdot e_\nu = e^i_\mu(x) e^j_\nu(x) \eta_{ij} \tag{4.6}$$

4.2 Local Poincaré Symmetry

Special relativity tells us that our experiments have no preferred frame. Furthermore, we can also deduce that our experiments have no preferred origin. For these reasons, a global Poincaré symmetry is physically inviting. Mathematically, as was shown in §2.5 and §2.6, a global Poincaré symmetry implies conservation of energy-momentum and angular momentum. It is therefore interesting to consider a gauge theory based on the group of Poincaré transformations. Specifically, we are interested in the physical ramifications of a Poincaré gauge theory (PGT). In this sense, we will develop a local theory based on Poincaré invariance in the opposite manner observed in most sources in the literature (see for example Ali et al., 2009, Kawai, 1994, Hehl et al., 1976).

[12] As a departure from §3.2, Latin indices are now endowed either a covariance or contravariance.

4.2.1 Global Poincaré Invariance

First consider a general Lagrangian as a function of the field and its conjugate variables.

$$\mathscr{L}(\psi, \partial_i\psi; \eta_{ij}) \tag{4.7}$$

A global Poincaré transformation has the following form

$$x^i \to x^i + \omega_j{}^i x^j + \widetilde{a}^i \tag{4.8}$$

where $\{\omega_{ij} = \omega_{[ij]}, \widetilde{a}^i\}$ constitute the 10 parameters of the Poincaré group. As discussed in §4.1 we would like to consider only field transformations as opposed to transforming our coordinate systems. As such, (4.8) may be expressed as a field transformation in holonomic

$$\psi(x) \to \psi'(x) = \psi(x) + \omega^{ij}\gamma_{ij}\psi(x) \tag{4.9}$$

or anholonomic coordinates–a more useful form for physical consequences:

$$\psi(x) \to \psi'(x) \equiv \mathrm{U}\psi(x) = \psi(x) + (\omega^{\mu\nu}\gamma_{\mu\nu} - a^\mu\partial_\mu)\psi(x) \tag{4.10}$$

where the $\gamma_{ij} \equiv M_{ij}$ are the Lorentz generators encountered in §2.6 and

$$a^\mu = \widetilde{a}^\mu + \omega_\nu{}^\mu \delta_i^\nu x^i \tag{4.11}$$

We may vary our Lagrangian (4.7) the same way as in §2.6 and find global conservation laws corresponding to a Poincaré invariance via Noether's theorem:

$$\begin{aligned}
\delta S &= \int d^4x \mathscr{L}(\psi', \partial_i\psi') - \int d^4x \mathscr{L}(\psi, \partial_i\psi) \\
&= \int d^4x \left[\left(\frac{\partial\mathscr{L}}{\partial\psi} - \partial_i \frac{\partial\mathscr{L}}{\partial(\partial_i\psi)} \right) \delta\psi + \partial_i \left(\frac{\partial\mathscr{L}}{\partial(\partial_i\psi)}\delta\psi + a^\mu\delta_\mu^i\mathscr{L} \right) \right] \\
&= \int d^4x \partial_i \left[\omega^{\mu\nu} \left(\frac{\partial\mathscr{L}}{\partial(\partial_i\psi)} \{\gamma_{\mu\nu} - \delta_\mu^i x_i \partial_\nu\}\psi \right) + a^\mu \left(\delta_\mu^i \mathscr{L} - \frac{\partial\mathscr{L}}{\partial(\partial_i\psi)}\partial_\mu\psi \right) \right] \\
&\equiv \int d^4x \left[\omega^{\mu\nu} \left(\partial_i \tau_{\mu\nu}{}^i - \mathrm{T}_{[\mu\nu]} \right) + \partial_i(a^\mu \mathrm{T}_\mu{}^i) \right]
\end{aligned} \tag{4.12}$$

Therefore we find the global conservation laws of angular momentum and energy-momentum respectively (as in §2)

$$\partial_i \tau_{\mu\nu}{}^i - \mathrm{T}_{[\mu\nu]} = 0 \tag{4.13}$$

$$\partial_i \mathrm{T}_\mu{}^i = 0 \tag{4.14}$$

The conserved quantities $\{\tau_{\mu\nu}{}^i, \mathrm{T}_\mu{}^i\}$ take on their expected definitions:

$$\tau_{\mu\nu}{}^i = -\frac{\partial\mathscr{L}}{\partial(\partial_i\psi)}\gamma_{\mu\nu}\psi \tag{4.15}$$

$$\mathrm{T}_\mu{}^i = \delta_\mu^i\mathscr{L} - \frac{\partial\mathscr{L}}{\partial(\partial_i\psi)}\partial_\mu\psi \tag{4.16}$$

We now proceed via a Yang-Mills-type approach to gauging the global Poincaré symmetry.

4.2.2 Local Poincaré Invariance

Following the procedure of §3, we first allow our field transformation (4.10) to vary in spacetime:

$$\psi'(x) = [1 + \omega^{\mu\nu}(x)\gamma_{\mu\nu} - a^\mu(x)\partial_\mu]\,\psi(x) \tag{4.17}$$

We can then see that (4.13) and (4.14) no longer hold. Instead, (4.12) now becomes

$$\delta S = \int d^4x \partial_i \left(\frac{\partial \mathscr{L}}{\partial(\partial_i \psi)} \delta\psi + a^\mu(x) \delta^i_\mu \mathscr{L} \right) \tag{4.18}$$

$$= \int d^4x \partial_i \left(\frac{\partial \mathscr{L}}{\partial(\partial_i \psi)} \left[\omega^{\mu\nu}(x)\gamma_{\mu\nu} - a^\mu(x)\partial_\mu \right] \psi + a^\mu(x) \delta^i_\mu \mathscr{L} \right) \tag{4.19}$$

$$= \int d^4x \left\{ \partial_i \widetilde{a}^\mu(x) \mathrm{T}_\mu{}^i - \partial_i \omega^{\mu\nu}(x) \left[\tau_{\mu\nu}{}^i + x_\nu \mathrm{T}_\mu{}^i \right] \right\} \tag{4.20}$$

which is clearly not invariant under (4.13) and (4.14). As with the Yang-Mills theory, we must modify our derivative to be covariant. However, our gauge group includes varying translations so we must also allow our "origin" to compensate (Hehl et al., 1976). These two statements can be mathematized by the following adjustments[13]

$$\partial_i \to D_i \equiv \partial_i + \Gamma_i{}^{\mu\nu}(x)\gamma_{\mu\nu} \tag{4.21}$$

$$\delta^i_\mu \to e^i_\mu(x) \tag{4.22}$$

where $\{\Gamma_i{}^{\mu\nu}(x), e^i_\mu(x)\}$ constitute the compensating gauge potentials of our theory. $e^i_\mu(x)$ is the same tetrad field we encountered in §4.1. Dropping the explicit reference to spacetime dependence, we can now modify (4.17) to be

$$\mathscr{U}\psi = \left[1 + \omega^{\mu\nu}\gamma_{\nu\mu} - a^\mu D_\mu\right]\psi \tag{4.23}$$

where

$$e^i_\mu D_i = D_\mu \tag{4.24}$$

Using (4.23), (4.21), and (2.41) we can describe the group structure of our modified Poincaré generators $\{D_i, \gamma_{\mu\nu}\}$ as

$$[\gamma_{\mu\nu}, \gamma_{\alpha\beta}] = g_{\alpha[\mu}\gamma_{\nu]\beta} - g_{\beta[\mu}\gamma_{\nu]\alpha} \tag{4.25}$$

$$[\gamma_{\mu\nu}, D_\alpha] = g_{\alpha[\mu}D_{\nu]} \tag{4.26}$$

$$[D_\mu, D_\nu] = e^i_\mu e^j_\nu \left(F_{ij}{}^{\alpha\beta}\gamma_{\beta\alpha} - T_{ij}{}^\alpha D_\alpha \right) \tag{4.27}$$

with

$$F_{ij}{}^{\mu\nu} = F_{ij}{}^{[\mu\nu]} \equiv 2\partial_{[i}\Gamma_{j]}{}^{\mu\nu} + 2\Gamma_{[i}{}^{\alpha\nu}\Gamma_{j]}{}^{\beta\mu}g_{\alpha\beta} \tag{4.28}$$

$$T_{ij}{}^\mu = T_{[ij]}{}^\mu \equiv 2D_{[i}e^\mu_{j]} \tag{4.29}$$

where (4.28) and (4.29) are the rotational and translational field strengths respectively (see Datta, 1971 for more details). (4.25) and (4.26) follow from the invariance of $SO(1,3)$ under (4.21) and (4.22).

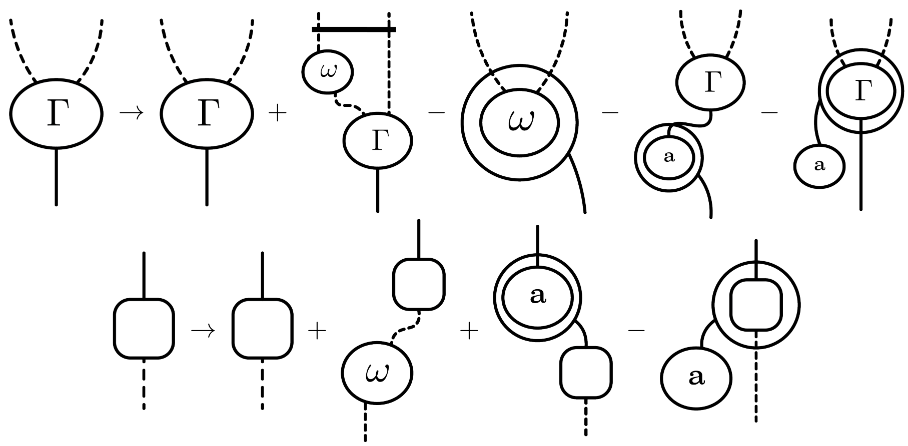
From (4.25), (4.26), (4.27), and (4.23) we can see

$$\delta\Gamma_i{}^{\mu\nu} = \omega^{[\mu}{}_\alpha\Gamma_i{}^{\alpha|\nu]} - \omega^{\mu\nu}{}_{,i} - a^j_{,i}\Gamma_j{}^{\mu\nu} - a^j\Gamma_{i,j}{}^{\mu\nu} \tag{4.30}$$

$$\delta e^i_\mu = \omega^\alpha{}_\mu e^i_\alpha + a^i_{,j}e^j_\mu - a^j e^i_{\mu,j} \tag{4.31}$$

[13]In anticipation of the geometric interpretation of the connection, we use the symbol Γ for our potential. However, this connection is not the Christoffel symbol. Rather, as we will see, this gauge potential corresponds to a torsionful connection, to be explored in the next section.

or diagrammatically



Now that we have the transformation laws of our gauge fields and their field strengths in hand, we seek the field equations of our theory. We can update (4.15) and (4.16) via our gauge field definitions

$$e\tau_{\mu\nu}{}^{i} = -\frac{\partial \mathscr{L}}{\partial(\partial_i \psi)} m_{\mu\nu}\psi \tag{4.32}$$

$$eT_{\mu}{}^{i} = e^{i}_{\mu}\mathscr{L} - \frac{\partial \mathscr{L}}{\partial(\partial_i \psi)} D_{\mu}\psi \tag{4.33}$$

where

$$e \equiv \det e^{i}_{\mu} = \sqrt{-g} = \sqrt{-\det g_{ij}} \tag{4.34}$$

is required to make our theory coordinate independent. For example, changing coordinates from the lab frame to an arbitrary frame we observe

$$g^{ij} = \xi^{i}{}_{a}\xi^{j}{}_{b}\eta^{ab} \tag{4.35}$$

and so the Jacobian of the coordinate transformation[14] $\xi^{i}{}_{a}$ is $\sqrt{-g}$.

From (4.13) and (4.14), we see that our gauge conservation laws read

$$D_i\left(e\tau_{\mu\nu}{}^{i}\right) = T_{[\mu\nu]} \tag{4.36}$$

$$D_i\left(eT_{\mu}{}^{i}\right) = e\left(F_{\mu i}{}^{\alpha\beta}\tau_{\alpha\beta}{}^{i} + F_{\mu i}{}^{\alpha}T_{\alpha}{}^{i}\right) \tag{4.37}$$

Although (4.36) is similar to (4.13), (4.37) tells us there is now an effect of the energy-momentum tensor. Namely, the gauge field strengths $\{F_{\mu i}{}^{\alpha\beta}, F_{\mu i}{}^{\alpha}\}$, are now coupled to our matter distribution sources $\{\tau_{\alpha\beta}{}^{i}, T_{\alpha}{}^{i}\}$. At an geometric level, these couplings have deep consequences. Indeed even comparing (4.30) and (4.31) with (3.18) exhibits the unconventional structure of external gauge theories (Hehl et al., 1976, Datta, 1971).

4.3 Einstein-Cartan-Sciama-Kibble Theory

As explored in §3, ultimately gauge theories modify the underlying geometry of the theory by introducing a gauge connection. In internally gauge theories, this procedure and the resulting theory are all described in flat spacetime. In externally gauge theories, spacetime itself is being gauged and so it is more natural to work in curved spacetime. To connect Poincaré gauge theory to the geometry of curved spacetime and outline its consequences, we now highlight some aspects of physics in a curved geometry.

[14] In Poincaré Gauge Theory, we use e instead of $\sqrt{-g}$ to emphasize the role of the vierbein.

4.3.1 Riemann-Cartan Geometry

For Poincaré gauge theory, we need a space that is linearly connected. Therefore, we introduce a manifold, M, with an affine connection, Γ. For example, a basis vector twists and turns when it is transferred infinitesimally in the manifold an amount [15].

$$\nabla_i e_j = \Gamma^k_{ji} e_k \tag{4.38}$$

We can then identify two important tensors associated with our manifold: curvature and torsion. As illustrated in Figure 2, curvature is the amount by which a vector turns when parallel transported along an infinitesimal rectangle. Torsion is the amount by which a vector twists when parallel transported along either the right or left side of the open quadrilateral as shown in Figure 2 (Misner et al., 1973). We quantify each of these quantities in terms of the affine connection as

$$R_{ijk}{}^l = 2\partial_{[i}\Gamma^l_{j]k} + 2\Gamma^l_{[i|m}\Gamma^m_{j]k} \tag{4.39}$$
$$S_{ij}{}^k = 2\Gamma^k_{[ij]} \tag{4.40}$$

where $R_{ijk}{}^l$ are the components of the Riemann curvature tensor, and $S_{ij}{}^k$ are the components of Cartan's torsion tensor.

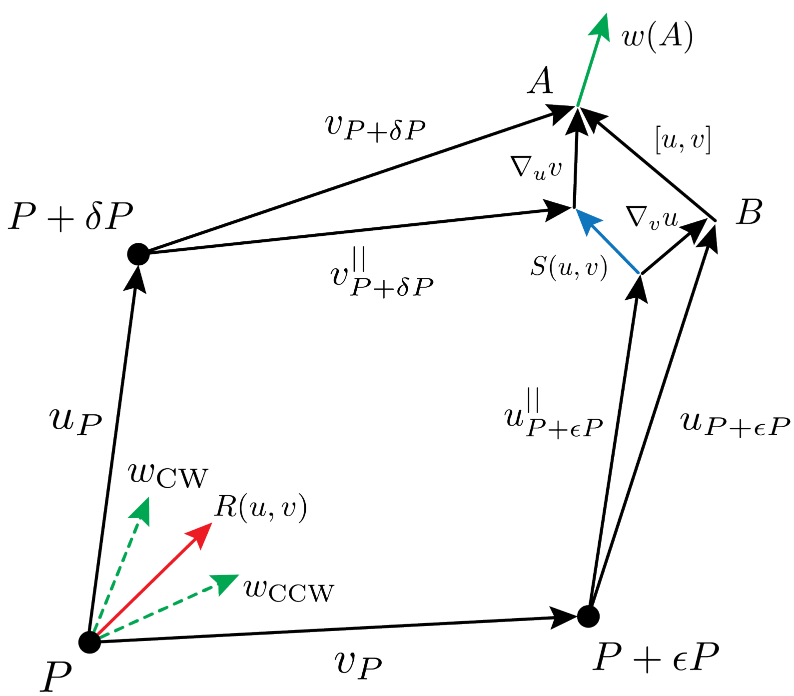


Figure 2: Torsion and curvature depicted heuristically in a 2D geometry. Torsion (blue) is the vectorial difference between the vector field u parallel-transported to $P + \epsilon P$ ($u^{||}_{P+\epsilon P}$) and the vector field v parallel-transported to $P + \delta P$ ($v^{||}_{P+\delta P}$). Curvature (red) is the vectorial difference between the vector field w (green) parallel-transported along the path $A \to P + \delta P \to P$ (denoted w_{CCW} for the counter-clockwise direction)and w parallel-transported along the path $A \to B \to P + \epsilon P \to P \to P$ (denoted w_{CW} for the clockwise direction) where ϵP and δP are infinitesimal.

To highlight the diffeomorphism invariance of our theory, we can also express these tensors as 2-forms

[15] Note there is a notational discrepancy between geometry and particle physics for covariant and exterior derivatives. In particle physics, the covariant derivative is denoted by D_i. In geometry, D is the *exterior* covariant derivative and ∇ is the covariant derivative. In align with the literature, D_i is used in §4.2.2 while ∇_i is used in §4.3.

via Cartan's structure equations (Misner et al., 1973)

$$R_{ij} = \mathrm{d}\Gamma_{ij} + \Gamma_{ik} \wedge \Gamma^k{}_j \tag{4.41}$$

$$S_i = \mathrm{d}e_i + \Gamma_{ik} \wedge e^k \tag{4.42}$$

where $\{\Gamma^k_{ij} dx^k, e^\mu_i dx^\mu\}$ are the connection 1-forms. Associated with Cartan's structure equations are the Bianchi identites

$$\mathrm{D}R_{ij} = 0 \tag{4.43}$$

$$\mathrm{D}S_i = R_{ij} \wedge e^j \tag{4.44}$$

where $\mathrm{D}A_i = \mathrm{d}A_i + \Gamma_{ik} \wedge A^k$ is the exterior covariant derivative with respect to the Lorentz connection 1-form. Also associated with our space is a metric defined by an interval expressed in a certain coordinate basis

$$ds^2 = -g_{ij}(x)dx^i dx^j \tag{4.45}$$

Usually we want the interval preserved under parallel transport. For example, in Minkowski space, the proper time is invariant under Lorentz transformations. This condition is quantified by the statement

$$\nabla g = 0 \tag{4.46}$$

From (4.46) we may write Γ in terms of S by first permuting indices

$$\nabla_i g_{jk} = g_{jk,i} - \Gamma^m{}_{ij} g_{km} - \Gamma^m{}_{ik} g_{jm} \tag{4.47}$$

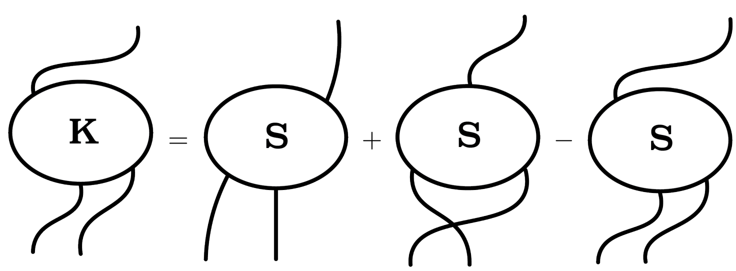
$$\nabla_k g_{ij} = g_{ij,k} - \Gamma^m{}_{ki} g_{jm} - \Gamma^m{}_{kj} g_{im} \tag{4.48}$$

$$-\nabla_j g_{ik} = -g_{ik,j} + \Gamma^m{}_{ji} g_{km} + \Gamma^m{}_{jk} g_{im} \tag{4.49}$$

Summing (4.47)-(4.49), we see

$$\begin{aligned} \Gamma^k{}_{ij} &= \mathring{\Gamma}^k{}_{ij} + \frac{1}{2}\left(S_{ij}{}^k - S_j{}^k{}_i - S^k{}_{ij}\right) \\ &= \mathring{\Gamma}^k{}_{ij} + K^k{}_{ij} \end{aligned} \tag{4.50}$$

where



(4.51)

is the contorsion tensor. It gives the deviation of the affine connection from the Christoffel symbols. A linearly connected geometry with a non-zero torsion is known as a Riemann-Cartan space (Hehl et al., 1976 and Diether III and Christian, 2020). The theory of gravitation in Riemann-Cartan space is known as Einstein-Cartan-Sciama-Kibble Theory (ECSK). In this way, we may see that our theory is a generalization of General Relativity (GR). If $S = 0$ then we recover the connection from GR. Furthermore, if we set $g = \eta$ then we recover Minkowski spacetime and the realm of Special Relativity. Figure 3 shows this generalization along with the conditions for each space.

In ECSK Theory, the Einstein tensor takes on its GR definition in terms of the Ricci tensor, R_{ij}, and the scalar curvature, R, albeit in terms of a torsionful connection:

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R \tag{4.52}$$

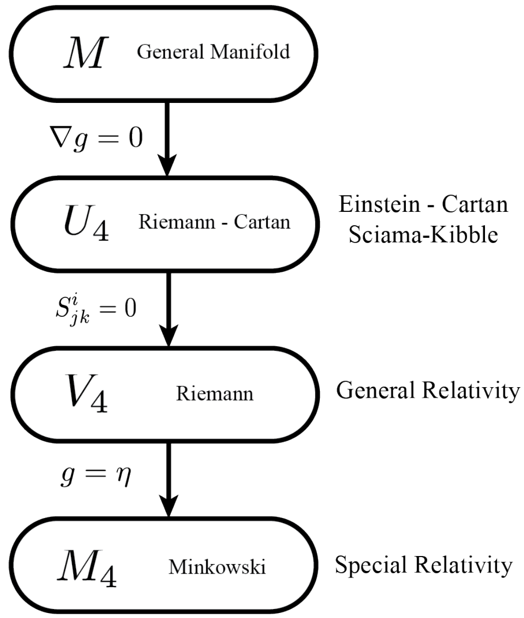


Figure 3: Various theories of spacetime dynamics along with their underlying geometries. Between each of the theories is the condition that must be satisfied to reduce one geometry to the next.

4.3.2 Conservation Laws and Field Equations

Now we have defined the geometry upon which ECSK theory is based, we proceed to derive conservation laws. Suppose we take as the Lagrangian of our theory the Einstein-Hilbert action (see Bertschinger, 2004)

$$S_{EH} = \int d^4x \; eR \tag{4.53}$$

so the total action is given by

$$\begin{aligned} S &= S_{\text{EH}} + S_M && (4.54) \\ &= \int d^4x \; eR + \int d^4x \mathscr{L}_M && (4.55) \end{aligned}$$

where S_M is some matter Lagrangian. The specific matter Lagrangian will depend on the particle content of our theory. Then we can define the metric energy-momentum tensor, the spin energy potential, and the spin energy potential with respect to the contorsion as

$$e\sigma^{ij} = 2\frac{\delta\mathscr{L}}{\delta g_{ij}} \tag{4.56}$$

$$e\mu_k{}^{ij} = \frac{\delta\mathscr{L}}{\delta S_{ji}{}^k} \tag{4.57}$$

$$e\tau^{ijk} = \frac{\delta\mathscr{L}}{\delta K^k{}_{ji}} \tag{4.58}$$

It follows $\tau^{ijk} = \mu^{[ji]k}$ (Hehl et al., 1976). We can also define a total energy momentum tensor

$$\Sigma^{ij} = \sigma^{ij} - \nabla_k \mu^{ijk} \tag{4.59}$$

Now if we require the first order variation in the total Lagrangian to vanish,

$$\delta\mathscr{L} = 0 \tag{4.60}$$

then we obtain the following conservation laws

$$\nabla_j \Sigma_i{}^j = 2\Sigma_k{}^j S_{ij}{}^k + \tau_{jkl} R_i{}^{ljk} \tag{4.61}$$

$$\nabla_k \tau_{ij}{}^k = \Sigma_{[ij]} \tag{4.62}$$

Upon comparison of (4.36), (4.37) and (4.61), (4.62) we see the connection between a Riemann-Cartan geometry and Poincaré gauge theory is illuminated. From the antisymmetry in (the non-Riemannian part of) Γ, we also see that in PGT the metricity condition is satisfied and so Poincaré gauge theory is set in a Riemann-Cartan spacetime. Correspondingly, we may identify the gauge field strengths as the torsion and curvature of spacetime

$$S_{ij}{}^k \leftrightarrow F_{\mu j}{}^{\alpha} \tag{4.63}$$

$$R^i{}_{jkl} \leftrightarrow F_{\mu i}{}^{\alpha\beta} \tag{4.64}$$

where equality is observed via the tetrad

$$S_{ij}{}^k = e^{\mu}_i e^k_{\alpha} F_{\mu j}{}^{\alpha} \tag{4.65}$$

$$R^i{}_{jkl} = e^i_{\mu} e^{\alpha}_k e^{\beta}_l F^{\mu}{}_{j\alpha\beta} \tag{4.66}$$

Now that we have identified spacetime torsion and curvature as the field strengths of PGT, we turn now to the field equations of ECSK theory to ascertain physical predictions of this theory. It much more convenient to derive the field equations in a geometric language (Datta, 1971). Therefore, we may rewrite (4.55) as

$$S_{\mathrm{ECSK}} = -\frac{1}{2\kappa}\int R^{ij} \wedge e^k \wedge e^l \epsilon_{ijkl} + \int \mathscr{L}_M \tag{4.67}$$

where $R_{ij} \equiv R^k{}_{ikj}$ is the Ricci tensor for a U_4 theory, $\kappa = 8\pi G$, and $\mathscr{L}_M$ is now understood to be a frame-invariant quantity[16]. If we now vary (4.67) with respect to the tetrad, e, we find

$$\delta S_e = \frac{-1}{4\kappa}\int \xi^k \wedge R^{ij} \wedge e^l \epsilon_{ijkl} - \int \xi \wedge \sigma \tag{4.68}$$

where ξ is an infinitesimal variation in the tetrad. Then requiring (4.68) to vanish gives the first field equation:

$$R^{ij} \wedge e^l \epsilon_{ijkl} = -2\kappa\sigma_k \tag{4.69}$$

If we vary (4.67) with respect to the Lorentz connection, Γ, we obtain

$$\begin{aligned}\delta S_\Gamma &= -\frac{1}{2\kappa}\int \left[R^{ij}(\Gamma+\zeta) - R^{ij}(\Gamma)\right] \wedge e^k \wedge e^l \epsilon_{ijkl} - \frac{1}{2}\int \zeta \wedge \tau \\ &= -\frac{1}{2\kappa}\int (\mathrm{D}\zeta)^{ij} \wedge e^k \wedge e^l \epsilon_{ijkl} - \frac{1}{2}\int \zeta \wedge \tau\end{aligned} \tag{4.70}$$

where

$$\begin{aligned}R(\Gamma+\zeta) &= \mathrm{d}\Gamma + \mathrm{d}\zeta + (\Gamma+\zeta)\wedge(\Gamma+\zeta) \\ &= R(\Gamma) + \mathrm{D}\zeta\end{aligned} \tag{4.71}$$

We may simplify the variation in the curvature in (4.70) by use of Stokes' theorem:

$$\begin{aligned}0 &= \int \mathrm{D}\left(\zeta^{ij} \wedge e^k \wedge e^l \epsilon_{ijkl}\right) \\ &= \int (\mathrm{D}\zeta)^{ij} \wedge e^k \wedge e^l \epsilon_{ijkl} - 2\int \zeta^{ij} \wedge (\mathrm{D}e)^k \wedge e^l \epsilon_{ijkl}\end{aligned} \tag{4.72}$$

[16] It is often useful to work in an orthonormal frame, so (4.67) contains only an orthonormal basis

By virtue of (4.42) and (4.72), we have

$$\delta S_\Gamma = -\frac{1}{2\kappa}\int \zeta^{ij} \wedge S^k \wedge e^l \epsilon_{ijkl} - \frac{1}{2}\int \zeta \wedge \tau \tag{4.73}$$

By requiring (4.73) to vanish, we arrive at the second field equation

$$S^k \wedge e^l \epsilon_{ijkl} = -\kappa \tau_{ij} \tag{4.74}$$

In component form, (4.69) and (4.74) read

$$R_{ij} - \frac{1}{2} R g_{ij} = \kappa \sigma_{ij} \tag{4.75}$$

$$S^i{}_{ai}\delta^k_b + S^k{}_{ba} + S^i{}_{ib}\delta^k_a = \kappa \tau_{ab}{}^k \tag{4.76}$$

Converting (4.69) and (4.74) to component notation is relatively straight-forward and is shown in Appendix B. In words the field equations state

Curvature = Energy-momentum

Torsion = Spin-angular-momentum

It is now clear that PGT is a generalization of General Relativity. To understand the physical implications of PGT, we first determine the uniqueness of each equation. In (4.74) there are 36 independent unknowns in Γ while there are only 16 independent equations. By contrast, (4.69) contains 24 equations and 24 unknowns. Therefore, only curvature and not torsion propagates. Outside of matter, torsion vanishes. This makes it extremely difficult to detect the effects of torsion. In vacuum, GR and PGT give the same results.

To see when torsion would have a noticeable effect, we first combine the two field equations, (4.69) and (4.74), to form a single equation. If we decompose (4.52) in a Riemannian and non-Riemannian part, we see

$$G_{ij} = \mathring{G}_{ij} + 2\left(\partial_{[m} K^m{}_{|i]j} + K^m{}_{[m|l} K^l{}_{|i]j}\right) - g_{ij} g^{ab}\left(\partial_{[m} K^m{}_{|a]b} + K^m{}_{[m|l} K^l{}_{|a]b}\right) \tag{4.77}$$

Since (4.69) represents an algebraic relation, we may "integrate out" torsion from the (4.77) to give a combined field equation

$$\mathring{G} \propto \kappa\left(\sigma + \kappa\tau^2 + \mathcal{O}(\kappa)^2\right) \tag{4.78}$$

(4.78) tells us spin corrections to GR are of order κ^2. Suppose we have a uniform mass density of $\rho_m = mn$ and spin density of $s = \frac{\hbar}{2}n$ where m is the mass of an individual particle. Then spin becomes important when, plugging in the neutron mass for the particles, (Hehl et al., 1976)

$$\rho_c = \frac{m^2}{\hbar^2 \kappa} = 10^{54} \text{ g cm}^{-3} \tag{4.79}$$

Since ρ_c is so large, it is very hard to detect the effects of torsion[17]. As of now, there is no experimental evidence that torsion is present (Shapiro, 2002). However PGT is more appealing than GR as it is a direct consequence of the Poincaré symmetry, which we expect to be respected in classical theories.

5 Conclusion

Symmetries imply important physical consequences in nature. Globally, symmetries imply there are conserved quantities. External global symmetries correspond to the conservation of measurable quantities such as momentum and energy, while internal global symmetries correspond to the conservation of more abstract Noether currents in the internal space. Upon gauging a global symmetry, we obtain physical interactions. For example, gauging a U(1) symmetry corresponds to electromagnetic interactions. The Standard Model is built on gauged internal symmetries. If we gauge an external symmetry, for example the Poincaré symmetry, we are forced to work in a curved space. In this curved space, a gauged Poincaré symmetry corresponds to the Einstein-Cartan-Sciama-Kibble theory of gravitation. Therefore, symmetries are extremely useful in producing physical predictions (conservation laws) and physical theories (gauge theories).

[17] Although they are hard to detect, it is important to include torsion-induced effects in situations with high spin densities. For example, models of the early universe and quantum gravity must include these effects (Datta, 1971).

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Appendices

A Conventions

Unless explicitly mentioned, we will work in natural units, $c = G = \hbar = 1$, when applicable. We will use the Einstein summation convention where all repeated indices are summed:

$$F^a{}_{ba} \equiv \sum_{i=0}^{N} F^i{}_{bi} \tag{A.1}$$

where $N = 3$ for spacetime indices. When we are working in Minkowski space, we will denote the metric by η_{ij}. In §2-§3, we adopt the "physical" convention

$$\eta_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

From §4 on, we use the "geometric" convention

$$\eta_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This choice reflects the geometric consequences of external gauge theories. An arbitrary metric we denote with g_{ij}. Matrix multiplication will sometimes be denoted by a dot:

$$A \cdot x \equiv A^i{}_j x^j \tag{A.2}$$

Unless otherwise stated, all fields are assumed to be functions of spacetime. To avoid clutter, derivatives are assumed to act on the object directly to their right unless indicated otherwise like so

$$\partial_\mu B^\nu A^\omega \equiv \frac{\partial B^\nu}{\partial x^\mu} A^\omega \tag{A.3}$$

$$\partial_\mu (B^\nu A^\omega) \equiv \frac{\partial (B^\nu A^\omega)}{\partial x^\mu} \tag{A.4}$$

Lie algebra-valued potentials are represented via a boldface font. For example

$$\mathbf{A}_\mu = t^a A^a_\mu \tag{A.5}$$

Define the following conventions for symmetrization and antisymmetrization of indices:

$$A^{[ad]}{}_{bc} \equiv \frac{1}{2}\left(A^{ad}{}_{bc} - A^{da}{}_{bc}\right) \tag{A.6}$$

$$A^{(ad)}{}_{bc} \equiv \frac{1}{2}\left(A^{ad}{}_{bc} + A^{da}{}_{bc}\right) \tag{A.7}$$

$$\tag{A.8}$$

so that

$$A^{ad}{}_{bc} = A^{[ad]}{}_{bc} + A^{(ad)}{}_{bc} \tag{A.9}$$

and define the following notational conventions for derivatives

$$A_{\mu,\nu} \equiv \frac{\partial A_\mu}{\partial x^\nu} \tag{A.10}$$

$$A_{\mu;\nu} \equiv \frac{\partial A_\mu}{\partial x^\nu} - A_\alpha \Gamma^\alpha_{\nu\mu} \tag{A.11}$$

$$A^\mu_{;\nu} \equiv \frac{\partial A^\mu}{\partial x^\nu} + A^\alpha \Gamma^\mu_{\nu\alpha} \tag{A.12}$$

In §4, quantities with respect to the Christoffel symbols are referred to using a circle above the symbol to distinguish from a general affine connection:

$$\mathring{\Gamma}^k_{ij} = \frac{1}{2} g^{ka} \left[2g_{a(i,j)} - g_{ij,a} \right] \tag{A.13}$$

We use the Feynman slash notation for objects contracted with the Dirac matrices:

$$\not{A} = \gamma^\mu A_\mu \tag{A.14}$$

The Dirac matrices obey the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{A.15}$$

The Pauli matrices, $\vec{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.16}$$

form a representation of SU(2)

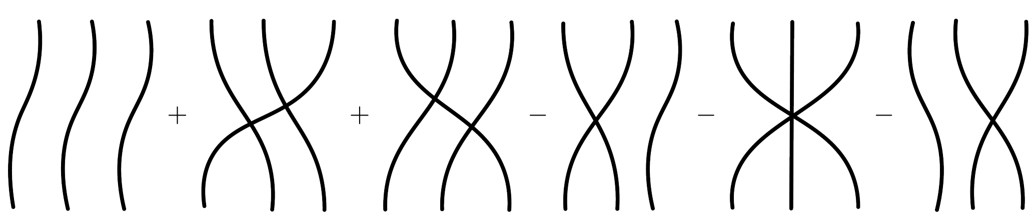
$$\left[\sigma^i, \sigma^j \right] = 2i\epsilon^{ijk} \sigma^k \tag{A.17}$$

B Einstein-Cartan-Sciama-Kibble Field Equations in Component Form

Starting with (4.69) we may take the Hodge star of both sides to find

$$\begin{aligned} \star \left(R^{ab} \wedge e^d \epsilon_{abcd} \right) &= \star \left(-2\kappa\sigma_c \right) \\ \star \left(\frac{1}{2} R^{ab}{}_{ij} e^i \wedge e^j \wedge e^d \epsilon_{abcd} \right) &= \star \left(-\frac{2\kappa}{3!} \sigma_c{}^a \epsilon_{arsd} e^r \wedge e^s \wedge e^d \right) \\ \frac{1}{2} R^{ab}{}_{ij} \epsilon_{abcd} \epsilon^{kijd} &= -\frac{2\kappa}{3!} \sigma_c{}^a \epsilon_{arsd} \epsilon^{krsd} \end{aligned} \tag{B.1}$$

Using $\epsilon_{arsd}\epsilon^{krsd} = -6\delta^k_a$ and $-\frac{1}{6}\epsilon_{abcd}\epsilon^{kijd}$ is given by



(B.2)

We can reduce (B.1) to

$$R_{ij} - \frac{1}{2} R g_{ij} = \kappa \sigma_{ij} \tag{B.3}$$

which has the same form as the Einstein Field Equations albeit with a connection containing a contribution from torsion. We apply the same procedure to (4.74):

$$\star \left(S^c \wedge e^d \epsilon_{abcd} \right) = \star \left(-\kappa \tau_{ab} \right) \tag{B.4}$$

$$\star \left(\frac{1}{2} S^c{}_{rs} e^r \wedge e^s \wedge e^d \epsilon_{abcd} \right) = \star \left(-\frac{\kappa}{3!} \tau_{ab}{}^c \epsilon_{crsd} e^r \wedge e^s \wedge e^d \right) \tag{B.5}$$

$$S^c{}_{rs} \epsilon_{abcd} \epsilon^{krsd} = \frac{2\kappa}{3!} \tau_{ab}{}^c \epsilon_{crsd} \epsilon^{krsd} \tag{B.6}$$

$$S^i{}_{ai} \delta^k_b + S^k{}_{ba} + S^i{}_{ib} \delta^k_a = \kappa \tau_{ab}{}^k \tag{B.7}$$

Therefore (B.3) and (B.7) constitute the field equations for ECSK in component form.

C Penrose Tensor Diagrams

To help visualize the transformation laws and definitions of some important results in §4, we utilize Penrose tensor diagrams (diagrams), developed by Roger Penrose in Penrose, 2005. The rules for constructing and reading diagrams are as follows (where dashed lines indicate local Lorentz indices and solid lines indicate global indices) [18].

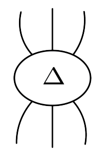
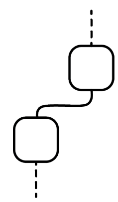
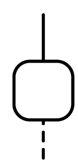
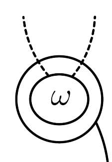
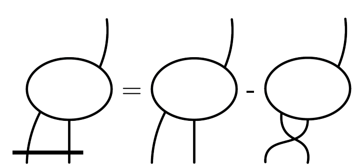
Rule	Object	Diagram
Tensors	Δ^{ijk}_{abc}	
Contraction	$e^i_\mu e^\nu_i$	
Kronecker Delta	δ^i_j	
Metric tensor	$g^{\mu\nu}$	
Tetrad field	e^i_μ	
Derivative	$\partial_\mu \omega^{ij}$	
Antisymmetrization	$2\ S_{[ij]}{}^k = S_{ij}{}^k - S_{ji}{}^k$	

Table 1: Rules for Penrose diagrams.

[18] Note we have modified his original rules to better suite our purposes

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